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Modelling the rip current flow on a quadratic beach profile

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ABSTRACT

*In this paper we develop an analytical theory for the interaction between waves and currents induced by breaking waves on time-scales longer than the individual waves. We employed the wave-averaging procedure that is commonly used in the literature. The near-shore zone is often characterized by the presence of breaking waves. Hence we develop equations to be used outside the surf zone, based on small-amplitude wave theory, and another set of equations to be used inside the surf zone, based on an empirical representation of breaking waves. Suitable matching conditions are applied at the boundary between the offshore shoaling zone and the near-shore surf zone. Essentially we derive a model for the interaction between waves and currents. Both sets of equation are obtained by averaging the basic equations over the wave phase. Thus the analytical solution constructed is a free vortex defined in both shoaling and surf zones. The surf zone solution is perturbed by a longshore component of the current. Thus the presence of the rip current cell combined with the longshore modulation in the wave forcing can drive longshore currents along the beach. The outcome, for our set of typical beach profile, is a description of rip current*s.

KEYWORDS*: *Wave-current interactions, surf zone, shoaling zone, matching conditions, wave-averaging, rip currents, radiation stress.

I. INTRODUCTION

Wave transformation in the surf zone is the dominant factor in the hydrodynamics of the nearshore circulation, and consequent sediment transport. A global description in terms of the spatial variation of such quantities such as wave action, wave action flux and wave radiation stress are the driving entities we have used to describe the generation by waves of mean currents in the nearshore zone. Studies on the interaction between waves and currents span nearly half a century. Mainly driven by a combination of engineering, shipping and coastal interests, there has been much research on shoaling nonlinear waves, on how currents affect waves and how waves can drive currents. This last aspect is the main concern in this paper.

The basis for this subject was laid down by Longuet-Higgins & Stewart(1960, 1961), who analyzed the nonlinear interaction between short waves and long waves (or currents), and showed that variations in the energy of the short waves correspond to work done by the long waves against the *radiation stress* of the short waves. In the shoaling zone this radiation stress leads to what are now known as wave setup and wave setdown, surf beats, the generation of smaller waves by longer waves, and the steepening of waves on adverse currents, see Longuet-Higgins & Stewart (1962, 1964). The divergence of the radiation stress was shown to generate an alongshore current by obliquely incident waves on a beach (Longuet-Higgins 1970).

The action of shoaling waves, and wave breaking in the surf zone, in generating a wave-generated mean sea-level is well-known and has been extensively studied, see for instance the monographs of Mei (1983) and Svendsen (2006). The simplest model is obtained by averaging the oscillatory wave field over the wave phase to obtain a set of equations describing the evolution of the mean fields in the shoaling zone based on small-amplitude wave theory and then combining these with averaged mass and momentum equations in the surf zone, where empirical formulae are used for the breaking waves. These lead to a prediction of steady set-down in the shoaling zone, and a set-up in the surf zone. This agrees quite well with experiments and observations, see Bowen et al (1968) for instance. However, these models assume that the sea bottom is rigid, and ignore the possible effects of sand transport by the wave currents, and the wave-generated mean currents. Hydrodynamic flow regimes where the mean currents essentially form one or more circulation cells are known as rip currents. These form due to forcing by longshore variability in the incident wave field, or the effect of longshore variability in the bottom topography (Kennedy 2003, 2005 ,Yu & Slinn 2003, Yu 2006 and others). They are often associated with significant bottom sediment transport, and are dangerous features on many surf beaches (Lascody 1998 & Kennedy 2005).

There is a vast literature on rip current due to wave-current interactions, see the recent works by (Horikawa 1978 , Damgaard *et al.* 2002, Ozkan-Haller & Kirby 2003 ,Yu & Slinn 2003 , Yu 2006, Falques, Calvete & Monototo 1998a and Falques *et al* 1999b, Zhang *et al* 2004 and others) and the references therein. Our purpose in this paper is to extend the wave-current interaction model examined in [5] to the quadratic depth.

In section 2 we record the usual wave-averaged mean field equations that are commonly used in the literature. In section 3 we introduce a description of the rip current formation and examine the consequences for both shoaling and surf zones. Then in section 4 we employ section 3 to extend our previous results see Osaisai, E. F (2013), now to include the rip currents behavior in a quadratic depth profile. We conclude with a discussion in section 5.

II. MATHEMATICAL FORMULATION

2.1 Wave field

In this section we recall the wave-averaged mean flow and wave action equations that are commonly used to describe the near-shore circulation (see Mei 1983 or Svendsen 2006 for instance). We suppose that the depth and the mean flow are slowly-varying compared to the waves. Then we define a wave-phase averaging operator $< f >= \bar{f}$, so that we can express all quantities as a mean field and a wave perturbation, denoted by a "tilde" overbar. For instance,

$$\zeta = \bar{\zeta} + \tilde{\zeta} . \tag{1}$$

where ζ is the free surface elevation above the undisturbed depth $h = h(x)$. Then outside the surf zone, the representation for slowly-varying, small-amplitude waves is, in standard notation,

$$\tilde{\zeta}(x,t) : a \cos \theta . \tag{2}$$

Here $a = a(x.t)$ is the wave amplitude and $\theta = \theta(x,t)$ is the phase, such that the wavenumber k , frequency Ω are given by

$$k = (k,l) = \nabla\theta \qquad \Omega = -\theta_t . \tag{3}$$

The local dispersion relation is

$$\Omega = \omega + k.U , \qquad \omega^2 = g\kappa \tanh \kappa H \tag{4}$$

$$where \qquad \kappa^2 = k^2 + l^2 .$$

Here $U(x,t)$ is the slowly-varying depth-averaged mean current (see below), and $H(x,t) = h(x) + \bar{\zeta}(x,t)$. To leading order, the horizontal and vertical components of the wave velocity field are respectively

$$\tilde{u} : \frac{k}{\kappa} a\Omega \frac{\cosh \kappa(z+h)}{\sinh \kappa H} \cos \theta , \qquad \tilde{w} : a\Omega \frac{\sinh \kappa(z+h)}{\sinh \kappa H} \sin \theta . \tag{5}$$

Importantly, note that we have ignored here any reflected wave field, which is assumed to be very weak when the bottom topography is slowly varying. The basic equations governing the wave field is then the kinematic equation for conservation of waves

$$k_t + \nabla\omega = 0 , \tag{6}$$

which is obtained from (3) by cross-differentiation, the local dispersion relation (4), and the wave action equation for the wave amplitude

$$A_t + \nabla \cdot (c_g A) = 0 . \tag{7}$$

Here $A = E/\omega$, where $E = ga^2/2$ is the wave energy per unit mass, and $c_g = \nabla_k \cdot \omega = U + c_g k/\kappa , (c_g = d\omega/d\kappa)$ is the group velocity. Using the dispersion relation (4) in (6) we get

$$k_t + c_g \cdot \nabla k = -\nabla_{ex} \Omega , \quad (8)$$

$$\Omega_t + c_g \cdot \nabla \Omega = \Omega_{et} . \quad (9)$$

Here the subscript "e" denotes the explicit derivative of $\Omega(k, x, t)$ with respect to either x or t, when the wavenumber k is held fixed. In this water case these explicit derivatives arise through the dependence of Ω on the mean height H and the mean current U.

2.2 Mean fields

The equations governing the mean fields are obtained by averaging the depth-integrated Euler equations over the wave phase. Thus the averaged equation for conservation of mass is

$$H_t + \nabla \cdot (HU) = 0. \quad (10)$$

Here $H = h + \bar{\zeta}$ where $h = h(x)$ is the time-independent undisturbed depth. For the velocity field we proceed in a slightly different way, that is we define

$$u = U + u' , \quad (11)$$

where we define U so that the mean momentum density is given by

$$M = HU = < \int_{-h}^{\zeta} u \, dz > , \quad (12)$$

But now we need to note that u' does not necessarily have zero mean, and that U and u are not necessarily the same. Indeed, from (11) and (12) we get that

$$u = U + < u' > , \quad and \quad < \int_{-h}^{\zeta} u' \, dz > = 0 .$$

But $u' = u + O(a^2)$, so that $< u' >$ is $O(a^2)$ and it follows that, correct to second order in wave amplitude,

$$M = Hu + M_w , \quad where \quad M_w = -H < u' > = < \zeta u(x, 0, t) > = \frac{E}{c} \frac{k}{\kappa} . \quad (13)$$

The term M_w in (13) is called the wave momentum.

Next, averaging the depth-integrated horizontal momentum equation yields (Mei 1983)

$$(HU)_t + \nabla .(HUU) = -\nabla . < \int_{-h}^{\zeta} u' u' + pI \, dz > + < p(z = -h) > \nabla h .$$

Next an estimate of the bottom pressure term is made by averaging the vertical momentum equation to get

$$< p(z = -h) > - g(\bar{\zeta} + h) = \nabla . < \int_{-h}^{\zeta} w u \, dz > + < \int_{-h}^{\zeta} w \, dz >_t .$$

For slowly-varying small-amplitude waves, the integral terms on the right-hand side may be neglected, and so $< p(z = -h) > \approx g(\bar{\zeta} + h)$. Using this in the averaged horizontal momentum equation, and replacing the pressure p with the dynamic pressure $q = p + (z - \bar{\zeta})$ yields

$$HU_t + \nabla .(HUU) = -\nabla .S - gH \nabla \bar{\zeta} \quad (14)$$

$$where \quad S = < \int_{-h}^{\zeta} [uu + qI] dz > - < \frac{g}{2} \tilde{\zeta}^2 > I . \quad (15)$$

Here S is the radiation stress tensor. In the absence of any background current, so that U is $O(a^2)$, we may use the linearized expressions (2, 5) to find that

$$S \approx c_g k \frac{E}{\omega} + E[\frac{c_g}{c} - \frac{1}{2}] I . \quad (16)$$

where the phase speed $c = \omega / \kappa$, correct to second order in the wave amplitude.

2.3 Shoaling zone

These equations hold in the shoaling zone outside the surf zone (defined below). In summary, the equations to be solved are that for the conservation of waves (6) combined with the dispersion relation (4), the wave action equation (7), the averaged equation for conservation of mass (10) and the averaged equation for conservation of horizontal momentum (14), where the radiation stress tensor is given by (16). In this shoaling zone, we assume that wave amplitudes are small, and that there is no background current. Then all mean quantities are $O(a^2)$, and in particular we can systematically replace H with h throughout these equations.

Next we shall suppose that $h = h(x)$ depends only on the offshore co-ordinate $x > 0$, where the undisturbed shoreline is at $x = 0$ defined by $h(0) = 0$. Further, in the near-shore region, including all of the surf zone, we shall assume that $h_x > 0$. Then we seek steady solutions of the equation set in which all variables are independent of the time t, and are also independent of the transverse variable y. It then follows from the mean mass equation (10) that HU is constant, and since $H = 0$ at the shoreline, it follows that we can set $U = 0$ everywhere. Then in the dispersion relation (4) $\Omega = \omega$. From the equation for conservation of waves (6) we see that the frequency ω and the transverse wavenumber l are constants, and the the offshore wavenumber k is then determined from the dispersion relation (4). As is well-known, it then follows that as $H \to 0$, $|k| \to \infty$, that is the waves refract towards the onshore direction, where we assume that the waves are propagating towards the shoreline so that $k < 0$. The wave action equation (7) reduces to Ec_g is constant. Near the shore, we can assume that the shallow water approximation holds and then $c_g \approx -(gh)^{1/2}$, so that

$$a^2 h^{1/2} \approx a_0^2 h_0^{1/2}, \tag{17}$$

where a_0 is the wave amplitude at a location offshore where $h = h_0$. The surf zone $x < x_b, h < h_b = h(x_b)$ can then be defined by the criterion that h_b is that depth where $a/h = A_{cr}$, defining an empirical breaking condition. A suitable value is $A_{cr} = 0.44$, see Mei (1983) or Svendsen (2006).

The last step is to find the wave set-up $\bar{\zeta}$ from the mean momentum equation (14), which here becomes

$$S_x + gH\bar{\zeta}_x = 0,, \tag{18}$$

$$where \qquad S = \frac{c_g}{c} E \cos^2\phi + \left(\frac{c_g}{c} - \frac{1}{2}\right) E.$$

Here ϕ is the angle between the wave direction and the onshore direction, and S is the "xx" component of the tensor S. As $h \to 0$, $\phi \approx 0, c_g \approx c, S \approx 3E/2$, and we recover the well-known result of a wave set-down in the shoaling zone

$$\bar{\zeta} = -\frac{a^2}{4h} = -\frac{a_0^2 h_0^{1/2}}{4h^{3/2}}. \tag{19}$$

Here we have assumed that $\bar{\zeta}$ is zero far offshore. Note that the first expression for $\bar{\zeta}$ does not need the use of the shallow water approximation, as shown by Longuet-Higgins and Stewart (1962).

2.4 Surf zone

In the surf zone $0 < x < x_b, 0 < h < h_b$, we make the usual assumption (see Mei (1983) for instance) that the breaking wave height $2a$ is proportional to the total depth H, so that

$$2a = \gamma H, \quad or \quad E = \frac{g\gamma^2 H^2}{8}, \tag{20}$$

Here the constant γ is determined empirically, and a typical value is $\gamma = 0.88.$. To determine the mean height $H = h + \bar{\zeta}$, we again use the mean momentum equation (14), but now assume that

$S = 3E/2 = \Gamma g H^2/2$ where $\Gamma = 3\gamma^2/8$, so that

$$\Gamma H H_x + H(H - h)_x = 0, \quad so \;\; that \quad H = H_b + \frac{h - h_b}{(1+\Gamma)}, \tag{21}$$

where the constant $H_b = h_b + \overline{\zeta}_b$ is determined by requiring continuity of the total mean height at $x = x_b$. Note that using (19)

$$H_b = h_b - a_0^2 h_0^{1/2}/4h_b^{3/2},$$

and since H_b must be positive, there is a restriction on either the deep-water wave amplitude a_0 or on the breaker depth h_b,

$$a_0^2/4h_0^2 < h_b^{5/2}/h_0^{5/2}. \tag{22}$$

Note that the expression (21) is valid for any depth $h(x)$, although in the literature it is often derived only for a linear depth profile $h = \alpha x$.

We are now in a position to determine the displaced shoreline $x = x_s$, defined by the condition that $H = 0$. That is, if $h_s = h(x_s)$ then $H = (h - h_s)/1 + \Gamma$, where

$$h_s = -\Gamma h_b - (1+\Gamma)\overline{\zeta}_b, \tag{23}$$

Note that to use the expression (23) it may be necessary to extend the definition $h(x)$ into $x < 0$. For instance for a linear beach, $h = \alpha x$, this is straightforward, but for a quadratic beach profile, $h = \beta x^2$, the extension for negative x should be $h = -\beta x^2$ say. Note that from (19),

$$\zeta_b = -\frac{a_0^2 h_0^{1/2}}{4h_b^{3/2}},$$

and, on combining this with the condition (22), it follows that the shoreline recedes (advances), that is $h_s < 0 (> 0)$ when

$$\frac{a_0^2}{4h_0^2} < \frac{\Gamma}{1+\Gamma}\frac{h_b^{5/2}}{h_0^{5/2}},$$

$$or \qquad \frac{\Gamma}{1+\Gamma}\frac{h_b^{5/2}}{h_0^{5/2}} < \frac{a_0^2}{4h_0^2} < h_b^{5/2}/h_0^{5/2}. \tag{24}$$

This anomalous result does seem only to have been recently noticed [5] [5] and [5]. Since there is an expectation that the shoreline should advance (see Dean and Dalrymple (2002) for instance), essentially it states that the present model is only valid for sufficiently small waves far offshore, defined by the first inequality in (24), which slightly refines the constraint (22). Next we can normalized the generalized bottom profile

$$H = h - h_s 1 + \Gamma. \tag{25}$$

so that it can be written in the form

$$\frac{H}{h_b} = \frac{1}{1+\Gamma}\left(\frac{h}{h_b} - \frac{h_s}{h_b}\right). \tag{26}$$

Thus all the depths profiles we have examined can be plotted as normalized functions.

2.5 Linear depth

Now for the linear depth, this is just, for our parameters $1/(1+\Gamma)(x/x_b - x_s/x_b)$ where

$$\frac{x_s}{x_b} = -\Gamma + (1+\Gamma)\frac{a_0^2}{4h_0^2}\frac{h_0^{5/2}}{h_b^{5/2}} \tag{27}$$

and so depends on the wave input a_0^2/h_0^2 and the ratio h_0/h_b. The normalized plots are shown in figure [1] below. It shows wave height for linear and quadratic beaches. Observe, in linear case the slope is, $\alpha/(1+\Gamma)$, and hence is smaller than the undisturbed slope, α. The plot is a function of x/x_b from x_s/x_b to 1 where we have evaluated x_s/x_b. Hence the plots depend on these two parameters, and we choose say $a_0/h_0 = 0.1$ and $h_b/h_0 = 1/4$ (gives $x_s < 0$), which gives $x_s/x_b = -0.2$.

2.6 Quadratic depth

The general bottom depth profile is given by equation (25). Our normalized parameters are $1/(1+\Gamma)(x\,|\,x\,|\,/x_b^2 - (x_s/x_b)^2)$. Thus equation (27) is the same as that above for the linear depth case, except that the left-hand side becomes

$$\pm\frac{x_s^2}{x_b^2} = -\Gamma + (1+\Gamma)\frac{a_0^2}{4h_0^2}\frac{h_0^{5/2}}{h_b^{5/2}}.$$

Here the depth is $\pm\beta x^2$ where the sign is for $x_s < 0$, or $x_s > 0$ which can be written as $\beta\,|\,x\,|\,x$. It follows that for the same parameters the right-side is again -0.2, and so $x_s/x_b = -0.45$. Figure [1] shows that they agree at $x = 0$ as the expression (25) already shows, but that the linear depth gives a greater setup in $x > 0$, but is weaker in the region $x < 0$.

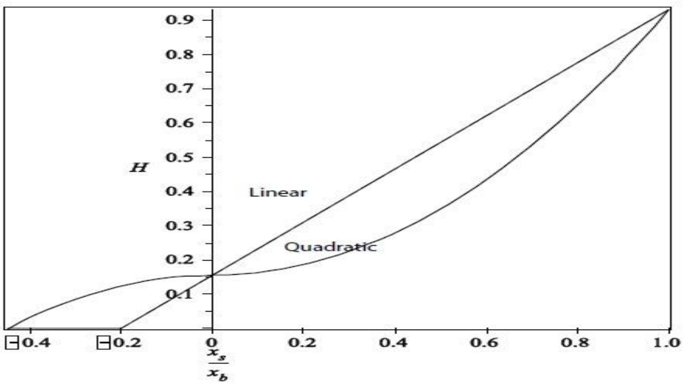

Figure 1: Plot of normalized wave height given by equation (25) for linear and quadratic profiles which depends on the ratios $a_0/h_0 = 0.1$ and $h_b/h_0 = 1/4$. The values of x ranges from $-0.2 < x/x_b < 1$ for the linear depth and $-0.45 < x/x_b < 1$ for the quadratic depth profile. Thus the graphs are plotted in the range $-0.45 < x/x_b < 1$.

III. THE GENERAL DESCRIPTION FOR THE RIP CURRENT FORMATION

Here we consider a steady-state model driven by an incident wave field which has an imposed longshore variability. The wave field satisfies equation (7) which in the present steady-state case reduces to

$$\frac{\partial}{\partial x}(Ec_g\cos\theta) + \frac{\partial}{\partial y}(Ec_g\sin\theta) = 0. \qquad (28)$$

Here we again assume that $h = h(x)$ and that consequently the frequency ω and the longshore wavenumber l are constants, while the onshore wavenumber K is then determined from equation (4). We have the wave energy E of the form

$$E = E_0(x)\cos Ky + F_0(x)\sin Ky + G_0(x), \qquad (29)$$

where the longshore period $2\pi/K$ is imposed. These equations in the shoaling zone yields

$$(E_0c_g\cos\theta)_x + KF_0c_g\sin\theta = 0 \qquad (30)$$

$$(F_0c_g\cos\theta)_x - KE_0c_g\sin\theta = 0 \qquad (31)$$

$$(G_0c_g\cos\theta)_x = 0. \qquad (32)$$

on collecting terms in $\cos Ky$, $\sin Ky$ and the constant term, which form three equations for E_0 , F_0 and G_0 . Equation (32) easily yields that $G_0 c_g \cos\theta = constant$. In shallow water, we may approximate by putting $c_g \approx gh^{1/2}$ and $\cos\theta \approx 1$, so that then $G_0 \approx constant \ /h^{1/2}$. For the remaining equations we can use Snell's law, $\sin\theta/c = \sin\theta_b/c_b = \alpha_b$ (the constant value, here evaluated at the breaker line), and the shallow-water approximation to get that

$$\{(E_0 c)_x)/c\}_x + K^2\alpha_b^2 E_0 c^2 = 0, \tag{33}$$

while although F_0 satisfies the same equation, once E_0 has been found, then F_0 is given by either (30) or (31). In practice, $Kc << 1$ and so approximately we can assume that $(E_0, F_0)c \approx constant$, the usual shallow-water expressions. Note that here $c \approx \sqrt{gh}$. In the surf zone, the expressions $E_0(x), F_0(x), G_0(x)$ is determined empirically.

Once the expression (29) has been determined, we may then substitute into the expressions (30,31 & 32) to obtain the radiation stress fields. Our aim here then is to describe how steady-state rip currents are forced by this longshore modulation of the incident wave field, especially in the surf zone.

The forced two-dimensional shallow water equations that we use here are characteristic of many nearshore studies (Horikawa 1978 , Damgaard *et al.* 2002, Ozkan-Haller & Kirby 2003 ,Yu & Slinn 2003 , Yu 2006, Falques, Calvete & Monototo 1998a and Falques *et al* 1999b, Zhang *et al* 2004 and others). Then, omitting the overbars as before, then equations (14) in the present steady-state case reduce to

$$H[U\partial U\partial x + V\partial U\partial y] = -gH\partial\zeta\partial x - [\tau_x], \tag{34}$$

$$H[U\partial V\partial x + V\partial V\partial y] = -gH\partial\zeta\partial y - [\tau_y], \tag{35}$$

where the stress terms are defined;

$$\tau_x = \partial S_{11}\partial x + \partial S_{12}\partial y \quad and \quad \tau_y = \partial S_{21}\partial x + \partial S_{22}\partial y. \tag{36}$$

Next we observe that equation (10) can be solved using a transport stream function $\psi(x, y)$, that is

$$U = -\frac{1}{H}\partial\psi\partial y \quad and \quad = \frac{1}{H}\partial\psi\partial x, \tag{37}$$

Next, eliminating the pressure, we get the mean vorticity equation

$$\psi_x\left(\frac{\Omega}{H}\right)_y - \psi_y\left(\frac{\Omega}{H}\right)_x = \left[\frac{\tau_x}{H}\right]_y - \left[\frac{\tau_y}{H}\right]_x \tag{38}$$

where Ω is define as

$$\Omega = V_x - U_y = \left(\frac{\psi_x}{H}\right)_x + \left(\frac{\psi_y}{H}\right)_y. \tag{39}$$

We shall solve this equation (38) in the shoaling zone $x > x_b$ and in the surf zone $x < x_b$, where as before $x = x_b$ is the fixed breaker line. It will turn out that the wave forcing occurs only in the surf zone, but continuity implies that the currents generated in the surf zone must be continued into the shoaling zone.

3.1 Shoaling zone

In $x > x_b$ we shall assume that $H \approx h$ as ζ is $O(a^2)$. Then we shall use the expressions [30 ,31] to evaluate the radiation stress tensor. For simplicity, we shall also use the shallow-water approximation that $c_g \approx c \approx \sqrt{gh}$, and so we get that

$$S_{11} = E(\cos^2\theta + \frac{1}{2}), S_{12} = S_{21} = E\sin\theta\cos\theta, S_{22} = E(\sin^2\theta + \frac{1}{2}) \quad (40)$$

These expressions are in principal known at this stage, and so we can proceed to evaluate the forcing term on the right-hand side of (38). To assist with this we recall Snell's law

$$\sin\theta = \sqrt{h}\sqrt{h_b}\sin\theta_b$$

where h_b and θ_b are the water depth and incidence angle at the breaker-line. Now the energy equation (28) has the approximate form

$$(Ec\cos\theta)_x + (Ec\sin\theta)_y = 0,$$

and using Snell's law, this can be written as

$$(E\cos^2\theta)_x + (E\sin\theta\cos\theta)_y + E\frac{c_x}{c} = 0,$$

$$\textit{and so} \quad \tau_x = \frac{1}{2}E_x - E\frac{c_x}{c}.$$

We can also deduce from (28) that

$$(E\sin\theta\cos\theta)_x + (E\sin^2\theta)_y = 0,$$

$$\textit{and so} \quad \tau_y = \frac{1}{2}E_y.$$

We can now evaluate the right-hand side of (38), and find that its identically zero,

$$[\frac{\tau_x}{h}]_y - [\frac{\tau_y}{h}]_x = 0.$$

Thus in the shoaling zone there is no wave forcing in the mean vorticity equation, although of course there will be a mean pressure gradient. However, this does not concern us since here our aim is to find only the flow field. Note that the result that there is no wave forcing in the vorticity equation does *not* need the specific form (29), and is based solely on the steady-state wave energy equation (28). The specific form (29) is only used in the surf zone.

With no forcing term, the vorticity equation (38) can be solved in the compact form, noting that we again approximate H with h,

$$\frac{\Omega}{h} = F(\psi). \quad (41)$$

But here $F(\psi) = 0$ from the boundary conditions in the deep water as $x \to \infty$, where the flow field is zero. Thus our rip current model has zero vorticity in the shoaling zone. It follows that we must solve the equation

$$\Omega = (\frac{1}{h}\psi_x)_x + (\frac{1}{h}\psi_y)_y = 0, \quad (42)$$

in $x > x_b$. Since $h = h(x)$ we can seek solutions in the separated form

$$\psi = X(x)Y(y) \quad (43)$$

with the outcome that

$$(\frac{X_x}{h})_x - \frac{K^2 X}{h} = 0, \quad Y_{yy} + K^2 Y = 0. \quad (44)$$

We note the separation constant $K^2 = 2\pi/L$ must not be zero, and is in fact chosen to be consistent with the modulation wavenumber of the wave forcing. Without loss of generality, we can choose

$$Y = \sin Ky . \tag{45}$$

For each specific choice of $h(x)$ we must then solve for $X(x)$ in $x > x_b$, with the boundary condition that $X \to 0$ as $x \to \infty$. We shall give details in the following subsections. Otherwise we complete the solution by solving the system (38) in the surf zone, and matching the solutions at the breakerline, $x = x_b$ where the streamfunction ψ must be continuous, and in order to have a continuous velocity field we must also have that ψ_x is continuous.

3.2 Surf zone

To make sense of wave forcing, we assume that the expression (29) holds in this region. The functions $E_0(x), F_0(x), G_0(x)$ are then determined empirically. To determine the wave forcing term in the mean vorticity equation (38) we shall assume that $\theta = \theta_b << 1$ so that, on using (??) and (40) we get that

$$\tau_x = \frac{3}{2} E_x , \quad \tau_y = \frac{1}{2} E_y .$$

Then (38) now becomes, where we again approximate H with $h(x)$,

$$\partial \psi \partial x \partial \tilde{\Omega} \partial y - \partial \psi \partial y \partial \tilde{\Omega} \partial x = \frac{E_{xy}}{h} + \frac{E_y h_x}{2h^2} = \frac{(h^{1/2} E_y)_x}{h^{3/2}}, \tag{46}$$

where here $\tilde{\Omega} = \Omega / h$ is the potential vorticity. Since the wave forcing is given by (29), that is

$$E = E_0(x) \cos Ky + F_0(x) \sin Ky + G_0(x), \tag{47}$$

we observe that the unmodulated term $G_0(x)$ plays no role here at all, although of course it will contribute to the wave setup. In order to match at $x = x_b$ with the expression (45) for the streamfunction in the shoaling zone, we should try for a solution of (46) of the form

$$\psi = F(x) \sin Ky + G(x), \quad in \quad x < x_b . \tag{48}$$

The matching conditions for the streamfunction and velocity field at the breakerline $x = x_b$ require that

$$F(x_b) = X(x_b), F_x(x_b) = X_x(x_b), G(x = x_b) = 0, G_x(x_b) = 0.$$

The expression (48) yields

$$\Omega = \tilde{F} \sin Ky + \tilde{G} \tag{49}$$

where $\tilde{F}$ and $\tilde{G}$ are differential operators where they are defined as;

$$\tilde{F} = \left(\frac{F_x}{h}\right)_x - \frac{K^2 F}{h} \tag{50}$$

$$\tilde{G} = Z_x , \; Z = \frac{G_x}{h} \tag{51}$$

From equation (46) we get a set of three equations that are used to determine the rip-current flow field in the surf zone. These are namely;

$$F_x \frac{\tilde{F}}{h} - F \left(\frac{\tilde{F}}{h}\right)_x = 0, \tag{52}$$

$$(G\frac{\tilde{F}}{h} - F(\frac{\tilde{G}}{h})_x) = \frac{(h^{1/2}F_0)_x}{h^{3/2}}, \tag{53}$$

$$(\frac{h^{1/2}E_0)_x}{h^{3/2}} = 0. \tag{54}$$

Equation (54) gives $E_0 : 1/h^{1/2}$, which is an unacceptable singularity as $h \to 0$. Hence we must infer that in the surf zone at least, $E_0 = 0$. The first of the three equations, that is (3.2a) suggests that

$$\frac{\tilde{F}}{h} = CF \; where C is a constant, \tag{55}$$

and the second (3.2b) yields that

$$F(CG - (\frac{\tilde{G}}{h})_x) = \frac{(h^{1/2}F_0)_x}{h^{3/2}} \tag{56}$$

The boundary conditions at $x = 0$ where $h = 0$ are that both mass transport fields U, V should vanish, that is from (37) $\psi = constant$ and $\psi_x / h = 0$, which implies that

$$F = F_x = 0, G = constant, \frac{G_x}{h} = 0, at \; x = 0. \tag{57}$$

As above there are also the matching conditions for both F and G separately at the breakerline, that is for F we have that

$$\frac{F_x(x_b)}{F(x_b)} = \frac{X_x(x_b)}{X(x_b)}, at \; x = x_b$$

where we note that here the right-hand side is a known quantity, depending only on K and x_b . Next we see that equation (55) reduces to

$$(\frac{F_x}{h})_x - \frac{K^2F}{h} = ChF \tag{58}$$

Together with the boundary conditions at $x = 0, x = x_b$ this is essentially an eigenvalue problem for $F(x)$ with eigenvalue C . In general it is solved approximately since we shall assume that $Kx_b << 1$. Once $F(x)$ is known we can solve (56), together with the appropriate boundary conditions to get $G(x)$ to complete the solution.

Note that the amplitude of $F(x)$ is an arbitrary constant in this solution, and so we can fix it by specifying its value at $x = x_b$ say. Indeed the solution we have constructed is essentially a free vortex defined by $X(x)\sin Ky$ in the shoaling zone $x > x_b$, and $F(x)\sin Ky$ in the surf zone $x < x_b$, perturbed by a longshore component $G(x)$ in the surf zone. Note that in the presence of the wave forcing, both F, G are non-zero, see (56). It is significant that unlike the longshore currents considered in the basic state which depend on an *ad hoc* frictional parametrization, the presence of the rip current cell combined with the longshore modulation in the wave forcing can drive a longshore current.

IV. APPLICATION TO A QUADRATIC DEPTH PROFILE

Osaisai, E.F (2013) examined the behavior of the rip-current for the case $h = \alpha x$. We here extend the case for which $h = \beta x^2$. In the basic state we show that the linear depth gives a greater setup in $x > 0$, but is weaker in the region $x < 0$, where both depths agreed at $x = 0$.

4.1 Shoaling zone

Now we let $h = \beta x^2$ and then equation (44) now admits the differential equation of the form

$$xX''(x) - 2X'(x) - K^2 xX(x) = 0 \tag{59}$$

whose solutions are explicitly the exponential functions of the form

$$X(x) = C_1 e^{Kx}(Kx - 1) + C_2 e^{-Kx}(Kx + 1).$$

From the behavior of the solutions as $x \to \infty$ we see that $C_1 = 0$ and so

$$X(x) = C_2 e^{-Kx}(Kx + 1). \tag{60}$$

Again note that the constant β does not appear in this solution.

4.2 Surf zone

Similarly, let $h = \beta x^2$ in (58), so that proceeding as for Osaisai, E.F (2013) we obtain the equation

$$F_{xx} - \frac{2}{x}F_x - K^2 F = A_0 \lambda x^7, \tag{61}$$

where now $\lambda = C\beta^2$, and as $x \to 0$, $F \approx A_0 x^3$. The solutions of the homogeneous equations are known, these are $j_1 = e^{-Kx}(Kx + 1)$ and $j_2 = e^{Kx}(Kx - 1)$ where j_1 and j_2 are linearly independent solutions. The general solution is given by

$$F(x) = A(x) j_1 + B(x) j_2 + C_1 j_1 + C_2 j_2, \tag{62}$$

where the Wronskian of j_1 and j_2 is $W = 2K^3 x^2$, and

$$A(x) = -\frac{A_0 \lambda}{2K^3}\int_0^x j_2 x^5 dx \approx \frac{A_0 \lambda x^6}{12K^3},$$

$$B(x) = \frac{A_0 \lambda}{2K^3}\int_0^x j_1 x^5 dx \approx -\frac{A_0 \lambda x^6}{12K^3}.$$

Thus A and B vanish at $x = 0$ as x^5 and so $C_1 = C_2$. Also the normalization of F as $x \to 0$ gives $C_1 = 3A_0/2K^2$.

Now to apply the boundary condition at $x = x_b$ which yields λ, we simplify the calculation by first approximating A and B as above for small x. The outcome is that

$$F \approx A_0 (x^3 + \frac{\lambda x^9}{36}).$$

The boundary condition at $x = x_b$ again yields an explicit equation for λ which can be simplified by the assumption that $Kx_b << 1$. Thus λ scales as x_b^{-6} which may not be so good an approximation. In spite of that the RHS may be approximated by $-K^2 x_b^2$. Hence to leading order

$$\lambda x_b^6 = -12. \tag{63}$$

This leads to the simple expression

$$F \approx A_0 x^3 (1 - \frac{x^6}{3x_b^6}), \tag{64}$$

and evidently $F_x \approx 0$ at $x = x_b$.

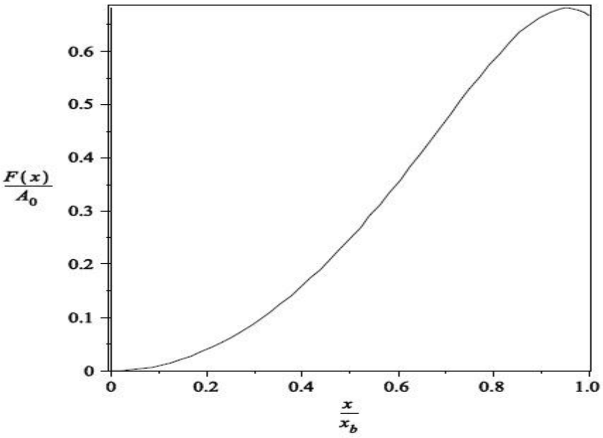

Figure 2: Plot of $F(x)/A_0$ against x/x_b , where A_0 is arbitrary as given by equation (64). Observe there is no dependence on the slope β but only a weak dependence on K . The value of $F(x)$ also reaches a maximum value of $F(x)/A_0 \approx 0.7$. This shows that irrespective of $h(x)$, the maximum value $F(x)$ can admit is ≈ 0.7 .

As in [5] we can now add correction terms, letting $\varepsilon = 1 + \lambda x_b^6/12$. Expanding A and B we find that

$$F(x) = A_0(x^3 - \frac{1}{4}K^4x^4 - \frac{1}{72}K^6x^6 + \lambda[\frac{1}{36}x^9 + \frac{2}{9}K^2x^{11}).$$

As before, we now find the leading order term for ε ,

$$\varepsilon = 1 + \lambda x_b^6/12 .$$

Finally we get that

$$F(x) = A_0x^3(1 - \frac{1}{4}K^4x - \frac{x^6}{3x_b^6} - \frac{Kx^6}{3x_b^5}). \tag{65}$$

Next, as before, we need to solve for $G(x)$ from (56). As above, we approximate $F = A_0x^3$, and also we use the empirical expression $F_0 = \gamma^2h^2/8$, see equation (20). Thus we get

$$Z_{xx} - \frac{2}{x}Z_x - C\beta^2x^4Z = -\frac{5g\gamma^2\beta^2}{8A_0} \tag{66}$$

Letting $u = x^3$ we get that

$$9Z_{uu} - \lambda Z = -\frac{5g\gamma^2\beta^2}{8A_0u^{\frac{4}{3}}}.$$

As before the dominant balance in the particular solution is between Z_{uu} and the right-hand side, so that $Z_p \approx constant \quad u^{2/3}$. We find that

$$Z_p = \frac{45\,g\gamma^2\beta^2x^2}{16\,A_0}. \tag{67}$$

As in the linear depth case, this have the form as $\lambda < 0$

$$Z_h = C_2 \sin \frac{\sqrt{\bar{\lambda}}\, u}{3} + C_3 \cos \frac{\sqrt{\bar{\lambda}}\, u}{3}.$$

Here we recall that $\bar{\lambda} = -\lambda$. The total solution is then

$$Z = Z_p + Z_h.$$

The boundary condition at $u = 0$ gives $C_3 = 0$. Again imposing the boundary conditions that $Z = 0$ at the breakerline $x = x_b$ gives,

$$Z = \frac{45\, g \beta^2 \gamma^2 x_b^2}{16\, A_0} \left(\frac{x^2}{x_b^2} - \frac{1}{\sin\, [2/\sqrt{3}\,]} \sin\, [\frac{2x^3}{\sqrt{3x_b^3}}] \right). \tag{68}$$

Finally we get G from $G_x = hZ$ and $G = 0$ at $x = x_b$,

$$G = \frac{45\, g \gamma^2 \beta^3 x_b^5}{16\, A_0} \left(\frac{x^5}{5x_b^5} + \frac{1}{2\sqrt{3}\, \sin\, [2/\sqrt{3}\,]} \cos\, [\frac{2x^3}{\sqrt{3x_b^3}}] - \frac{1}{5} - \frac{1}{2\sqrt{3}\, \tan\, [2/\sqrt{3}\,]} \right). \tag{69}$$

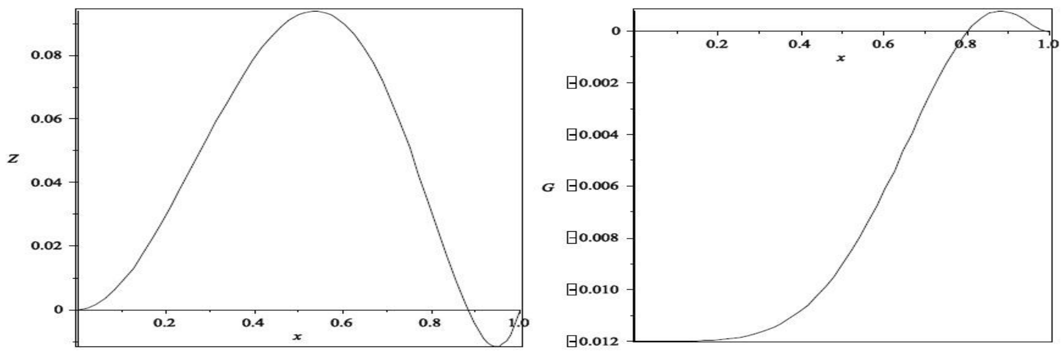

Figure 3: Depicts the plots of normalized $Z(x)$ and $G(x)$ given by equations (68) and (69) where each is normalized by $45\, g\beta^2\gamma^2 x_b^2/16\, A_0$ and $45\, g\beta^2\gamma^2 x_b^5/16\, A_0$ respectively with A_0 arbitrary. We observe as in [5], here too as depicted in the figure, there is a small region of reversed flow near the breaker line.

The combined expressions (60, 64, 69) complete the solution, where we recall that the constant C is given by (63) (since $\lambda = C\beta^2$), or their respective higher-order corrections. Now the amplitude of $F(x)$ at $x = x_b$ is given by

$$F(x_b) = C_2 e^{-Kx}(Kx + 1). \tag{70}$$

On using the approximation $Kx_b << 1$, and the approximate expression (64), this reduces to

$$F(x_b) = \frac{2A_0 x_b^3}{3} = C_2.$$

The rip-current system contains a free parameter A_0 or its equivalent. We choose to define this free parameter to be the value of $F(x_b)$ and normalize the full solution by this value. Thus we get from (43, 45) in $x > x_b$, and (48, 64, 69) in $x < x_b$ that the normalized streamfunction ψ_n is given by

$$\psi_n = \frac{X(x)}{X(x_b)} \sin(Ky), \quad for \quad x > x_b, \tag{71}$$

$$\psi_n = \frac{F(x)}{F(x_b)} \sin Ky + R\frac{G(x)}{G(0)}, \quad for \quad 0 < x < x_b. \tag{72}$$

Here again $R = G(0)/F(x_b)$ is a free parameter. From (64, 69) we find that here

$$R = \frac{135\, g\beta^3\gamma^2 x_b^2}{32\, A_0^2}\left(\frac{1}{2\sqrt{3}\sin[2/\sqrt{3}}-\frac{1}{5}-\frac{1}{2\sqrt{3}\tan[2/\sqrt{3}]}\right). \tag{73}$$

Note that again $R < 0$, and that $|R|$ increases as the wave forcing increases, or as the curvature β increases, or as the depth βx_b^2 at the breaker line increases. In order to estimate typical values for R we again note that from (64) the longshore velocity field in the "$\sin(Ky)$"-component scales as $V_c = A_0/\beta x_b$, while the longshore component then scales with RV_c. Taking account of the actual numerical values in the expressions given above, we find that a suitable values are $R \approx -0.1$. Plots of ψ_n are shown in figure [4 & 5] for same values of R as in the linear case, and again $Kx_b = 0.2$.

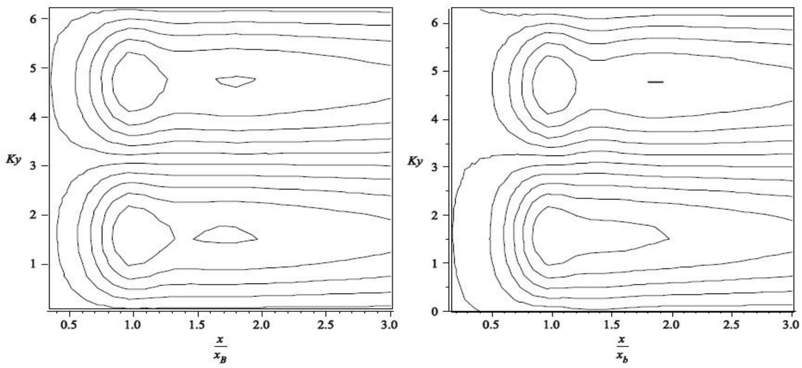

Figure 4: Plot of the rip current streamlines for a quadratic depth profile, given by equation (72) where $F(x)$ and $G(x)$ are equations (64) and (69) respectively for $R = -0.02$ in the left panel and $R = -0.1$ in the right panel.

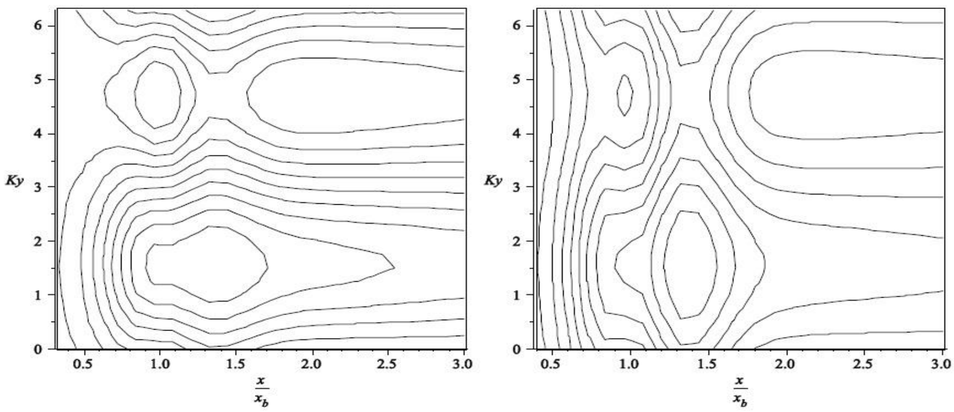

Figure 5: As for figure 4 but $R = -0.5$ in the left panel and $R = -2$ in the right panel.

Overall these plots show the same kind of behaviour as those for the linear depth profile see [5].

However, the major difference is that the flow in the surf zone is rather weaker, and so the vortex centre is slightly further offshore.

V. CONCLUSION

We described qualitative solutions for rip currents which are essentially free vortices in both zones. The free vortex in the surf zone is perturbed by a longshore modulation in the wave forcing. Rip current cell combining with the longshore modulation in the wave forcing can drive longshore currents along the beach. Thus the dynamics of the shoaling zone is only dependent on the state-state wave energy equation.The wave forcing in the surf zone sets the wave activities different from those of the shoaling zone. To determine wave forcing in the mean vorticity equation we assume that the wave angle becomes smaller. We also note here that the component of the radiation stress in the y momentum remains unchanged across the entire flow domain. This shows that it is only the x component of the radiation stress that play a leading role in the wave forcing. However, wave forcing encountered in the surf zone has an unmodulated term that *does not play a role* in the vorticity equation but only contribute to wave setup.To ensure continuity of the streamfunctions in the shoaling zone we match the solution at the breakerline by a matching condition with appropriate boundary conditions. Thus the rip currents solution in the surf zone is provided by the terms in the matching condition. The terms in the matching condition has a cross-shore width and a modulated longshore component. The cross-shore width was determined by the application of perturbation method and variation of parameter. It would to interesting to examine the effect of friction on the rip currents.

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